

DTP–MSU 95/39
October 1995

Geroch–Kinnersley–Chitre group for Dilaton–Axion Gravity¹

D.V. Gal'tsov

Department of Theoretical Physics, Moscow State University,
Moscow 119899, Russia

Abstract

Kinnersley–type representation is constructed for the four–dimensional Einstein–Maxwell–dilaton–axion system restricted to space–times possessing two non–null commuting Killing symmetries. New representation essentially uses the matrix–valued $SL(2, R)$ formulation and effectively reduces the construction of the Geroch group to the corresponding problem for the vacuum Einstein equations. An infinite hierarchy of potentials is introduced in terms of 2×2 real symmetric matrices directly generalizing the scalar hierarchy of Kinnersley–Chitre known for the vacuum Einstein equations.

PASC number(s): 97.60.Lf, 04.60.+n, 11.17.+y

¹A talk given at the International Workshop, “Quantum Field Theory under the Influence of External Conditions”, Leipzig Germany, 18–22 September 1995, published in the Proceedings, M. Bordag (Ed.), B.G. Teubner Verlagsgessellschaft, Stuttgart–Leipzig, 1996, pp. 228–237.

“Geroch–Kinnersley–Chitre group” is a popular name for the infinite symmetry arising in the Einstein and Einstein–matter theories enjoying a complete integrability property after reduction to two dimensions via imposition of a rank two Abelian isometry group on the four-dimensional space-time. An original observation by Geroch [1], essentially based on the earlier works of Ehlers [2] and Harrison [3], was then developed by Kinnersley [4] and Kinnersley and Chitre [5]. Important insights in establishing the integrability of the two-dimensional Einstein and Einstein–Maxwell (EM) theories were due to the works of Neugebauer and Kramer [6], [7] in which the relevance of the three-dimensional σ –models on symmetric spaces has been discovered. A concise complex potential representation given by Ernst [8] for both Einstein and Einstein–Maxwell theories provided new structures particularly useful in this research. Different proofs of the complete integrability of two-dimensional reductions of the Einstein and EM systems were given in the late 70–ths by Harrison [9], Maison [10], Belinskii and Zakharov [11], and Hauser and Ernst [12]. These alternative formulations as well as some other approaches [13], [14] were compared and extended by Cosgrove [15]. Extensive application of related methods in General Relativity was very fruitful in the search of exact classical solutions [16] (for more recent development and review see [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]).

In more general and modern language, Geroch group may be seen as an infinite-dimensional symmetry acting on harmonic maps of Riemann surfaces into symmetric spaces, the associated methods being related to the study of deformation of harmonic maps. In the gravity context such maps are connected to the three-dimensional σ –models to which the Einstein and some Einstein–matter systems are reduced for space-times admitting a non-null Killing vector field. If the corresponding target space is a symmetric riemannian space, its finite isometries undergo infinite affine extension when one goes to two dimensions. Some other physically interesting systems possess such a property, the most notable examples being the Kaluza–Klein [6], [27], [28] and certain supergravity models [29], [30], [31].

It is worth noting that similar infinite symmetries are encountered in a different class of theories, two-dimensional conformal field models, in particular, in the theory of superstrings. In the low energy limit such theories lead to the gravity–matter systems which may have in their turn the associated infinite symmetries. The investigation of links between two theories seems to be a promising field of research. Here we discuss Geroch–type symmetries found recently in the heterotic string low energy effective theory [32], [33], [34]. In the bosonic sector of this theory one finds the Einstein gravity coupled to massless vector and scalar fields. The simplest model of this kind incorporating basic features of the full effective action — “dilaton–axion gravity”, or Einstein–Maxwell–dilaton–axion (EMDA) system — can be formulated directly in four dimensions where it includes one $U(1)$ vector and two scalar fields coupled in such a way that the theory possesses non-abelian $SL(2, R)$ duality symmetry.

To begin, let us review briefly how Geroch symmetry emerges in the vacuum Einstein theory. Consider a four-dimensional manifold admitting a two-parameter group of motions generated by the Abelian algebra of Killing vectors K_A , $A = 1, 2$. In the adapted coordinates $K_A = \partial_A$ the interval can be written as

$$ds^2 = f_{AB}dx^A dx^B - h_{MN}dx^{M+2} dx^{N+2}, \quad (1)$$

where $A, B, M, N = 1, 2$ and f_{AB} , h_{MN} are real symmetric 2×2 matrices depending only on x^3 , x^4 . In the case of stationary axisymmetric metrics which we choose here for definiteness

a particular gauge (Lewis–Papapetrou) is appropriate:

$$f_{11} = f, \quad f_{12} = -f\omega, \quad f_{22} = f\omega^2 - \rho^2/f, \quad (2)$$

and

$$h_{11} = h_{22} = e^{2\gamma} f^{-1}, \quad (3)$$

where f , ω , γ are functions of $x^3 = \rho$, $x^4 = z$. The resulting two-dimensional equations can be conveniently written down using the Hodge-conjugated operators

$$\nabla = \partial_{N+2} = (\partial_z, -\partial_r h o), \quad \tilde{\nabla} = \tilde{\partial}_{N+2} = (-\partial_\rho, \partial_z). \quad (4)$$

The vacuum Einstein equations lead to the following system of equations for f_{AB}

$$\nabla (\rho^{-1} f^{AB} \nabla f_{BC}) = 0, \quad (5)$$

where rising and lowering of indices is effected using the alternating symbol

$$\epsilon_{AB} = -\epsilon^{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (6)$$

Once f_{AB} is found, the function γ can be constructed by solving a simple system of partial differential equations of the first order (what remains true for the EMDA system too, so we will not concentrate on this problem here).

These equations can be shown to possess two finite symmetry groups. The first is the so-called Matzner-Misner $SL(2, R)$ group G

$$x^A \rightarrow G^A{}_B x^B, \quad \det G = 1, \quad (7)$$

under which f_{AB} transforms as the $SL(2, R)$ tensor

$$f_{AB} \rightarrow G_A{}^C G_B{}^D f_{CD}. \quad (8)$$

The second, known as the Ehlers group H (also $SL(2, R)$), acts on the initial metric variables in a non-local way. To make this explicit one introduces a twist potential χ , the existence of which is implied by the 11-component of the Eq. (5), via duality relation

$$\tilde{\nabla} \chi = \rho^{-1} f^2 \nabla \omega. \quad (9)$$

The set of variables f , χ gives rise to another real symmetric matrix m_{AB} ,

$$m_{11} = f^{-1}, \quad m_{12} = \chi f^{-1}, \quad m_{22} = \chi^2 f^{-1} + f, \quad (10)$$

in terms of which the equivalent to (5) set of equations reads

$$\nabla (\rho m^{AB} \nabla m_{BC}) = 0. \quad (11)$$

This tensor transforms nonlinearly under the Ehlers group consisting of a gauge ($\chi \rightarrow \chi + \lambda_g$), scale ($f, \chi \rightarrow e^{2\lambda_s} f, e^{2\lambda_s} \chi$) and a proper Ehlers transformation, which can conveniently be expressed in terms of the Ernst potential $E = if - \chi$ as

$$E \rightarrow \frac{E}{1 + \lambda_E E}. \quad (12)$$

Acting by these transformations on some solution to the Eqs. (5) one obtains new solutions. Now, if G -covariance is taken into account, one can construct new symmetries by conjugation of H with G : GHG^{-1} . Repeating this operation one is led to an infinite-dimensional group

$$K = \dots H \times G \times H \times G \times H, \quad (13)$$

which is the Geroch–Kinnersley–Chitre group. It can be realised on an infinite hierarchy of potentials which may be introduced via the G -covariant dualization procedure [4].

The same reasoning was shown to hold for the EM system (Kinnersley and Chitre [5]) where the hierarchy is more complicated and involves some additional structures. Remarkably, the EMDA system turns out to be in this respect simpler than the EM one, and involves only the structures typical for the vacuum Einstein equations.

Consider the EMDA action containing a metric $g_{\mu\nu}$, a $U(1)$ vector field A_μ , a Kalb–Ramond antisymmetric tensor field $B_{\mu\nu}$, and a dilaton ϕ in four dimensions

$$S = \frac{1}{16\pi} \int \left\{ -R + \frac{1}{3} e^{-4\phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 2\partial_\mu\phi\partial^\mu\phi - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right\} \sqrt{-g} d^4x, \quad (14)$$

where

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - A_\mu F_{\nu\lambda} + \text{cyclic}, \quad (15)$$

and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. After reduction to two dimensions the set of dynamical quantities, in addition to metric variables described above, contains two components of the 4-potential

$$A_B = \frac{1}{\sqrt{2}}(v, a), \quad (16)$$

one non-trivial component of the Kalb–Ramond field

$$B_{12} = b, \quad (17)$$

and the dilaton ϕ . The corresponding system in presence of one Killing vector field was studied in [32] and in the two-dimensional case in [33] in G -noncovariant way. Here we give G -covariant formulation which opens a way to construct the Geroch–Kinnersley group explicitly. From the results of [35] it is clear that the underlying H group is a symplectic group $Sp(4, R)$, which fits well with the $SL(2, R)$ structure of the group G . One can find G -covariant description of the equations of motion simply by introducing an additional 2×2 matrix structure into the Kinnesley formalism described above.

First we introduce the following two 2×2 real symmetric matrices instead of the one-component quantities f and ω above:

$$P = \begin{pmatrix} f - e^{-2\phi} v^2 & -e^{-2\phi} v \\ -e^{-2\phi} v & -e^{-2\phi} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega & -q \\ -q & qv - b \end{pmatrix} \quad (18)$$

where $q = a + \omega v$. Using them as building blocks, one can now construct a G -tensor

$$F_{AB} = \begin{pmatrix} P & -P\Omega \\ -\Omega P & \Omega P \Omega - \rho^2 P^{-1} \end{pmatrix}. \quad (19)$$

Raising and lowering the indices now has to be performed with the matrix-valued alternating tensor

$$\epsilon_{AB} = -\epsilon^{AB} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (20)$$

Note that the inverse tensor

$$F^{-1}_{AB} = \rho^{-2} F^{AB} = \rho^{-2} \begin{pmatrix} \rho^2 P^{-1} - \Omega P \Omega & -\Omega P \\ -P \Omega & -P \end{pmatrix}. \quad (21)$$

In terms of this matrix the dynamical equations of motion assume essentially the same form as (5):

$$\nabla (\rho^{-1} F^{AB} \nabla F_{BC}) = 0. \quad (22)$$

Four 2×2 equations here are not algebraically independent. As two independent equations the 11 and 22 ones can be conveniently chosen

$$\nabla (\rho^{-1} P \nabla \Omega P) = 0, \quad (23)$$

$$\nabla (\rho \nabla P P^{-1} + \rho^{-1} P \nabla \Omega P \Omega) = 0, \quad (24)$$

The 22 component is related to (24) by transposition, while the 21 one is satisfied automatically due to both (23) and (24).

Ehlers group for the EMDA system was first described in [32] as some ten-parameter semisimple Lie group and then identified with $Sp(4, R)$ in [33]. It acts non-linearly on the set of six variables consisting of three initial variables f, v, ϕ and three dualized variables χ, u, κ . Here κ is the pseudoscalar Pecci–Quinn counterpart to the Kalb–Ramond field

$$H^{\mu\nu\lambda} = \frac{1}{2} e^{4\phi} E^{\mu\nu\lambda\tau} \frac{\partial \kappa}{\partial x^\tau}, \quad (25)$$

u is the magnetic potential related to the Maxwell tensor as

$$e^{-2\phi} F^{N2} + \kappa \tilde{F}^{N2} = \frac{f^2 e^{-2\gamma}}{\sqrt{2}\rho} \tilde{\partial}_{N+2} u, \quad (26)$$

while χ is the twist potential which now is the solution of the equation

$$\nabla \chi = u \nabla v - v \nabla u - \rho^{-1} f^2 \tilde{\nabla} \omega. \quad (27)$$

From six real variables entering the problem as a set of scalar fields one can build three complex potentials of the Ernst type

$$\Phi = u - zv, \quad E = if - \chi + v\Phi, \quad z = \kappa + ie^{-2\phi}. \quad (28)$$

It is worth noting that neither Φ , nor E reduce to the original Ernst potential for Einstein–Maxwell system. However their role in formulation of integrable system of equations is very similar to that of the original Ernst potentials (more detailed discussion can be found in [35]). Now we have one more complex variable (complex dilaton–axion field z) to describe the system. However the theory has additional symmetries which effectively reduce its complexity. One can show, in particular, that the whole theory (not only with two, but also with one Killing symmetry imposed on a four-dimensional manifold) is invariant under a discrete transformation

$$z' = E, \quad E' = z, \quad \Phi' = \Phi, \quad (29)$$

i.e. an interchange of z and E . Within the EMDA system the complex dilaton–axion field and the Ernst variable form quite similar algebraic structures.

New dualized variables can be combined within the 2×2 matrix Q related to Ω by matrix dualization [36]

$$\nabla Q = -\rho^{-1} P \tilde{\nabla} \Omega P. \quad (30)$$

Explicitly it reads

$$Q = \begin{pmatrix} wv - \chi & w \\ w & -\kappa \end{pmatrix}, \quad (31)$$

where $w = u - \kappa v$. Using P, Q pair one can now build 4×4 matrix transforming as an $Sp(2, R)$ matrix-valued object which is a direct analog of the matrix m_{AB} (10)

$$M_{AB} = \begin{pmatrix} P^{-1} & P^{-1}Q \\ QP^{-1} & P + QP^{-1}Q \end{pmatrix}. \quad (32)$$

In new terms the equations of motion read

$$\nabla (\rho M^{AB} \nabla M_{BC}) = 0. \quad (33)$$

The H -transformations are most conveniently described in terms of the complex combination

$$Z = Q + iP = \begin{pmatrix} E & \Phi \\ \Phi & -z \end{pmatrix}. \quad (34)$$

They consist of the shift on the constant real symmetric matrix

$$Z \rightarrow Z + R, \quad R = \begin{pmatrix} \lambda_g & \lambda_m \\ \lambda_m & -\lambda_{d_1} \end{pmatrix}, \quad (35)$$

matrix “scale” transformation

$$Z \rightarrow S^T Z S, \quad S = \begin{pmatrix} e^{\lambda_s} & \lambda_{H_1} \\ -\lambda_e & e^{\lambda_{d_3}} \end{pmatrix}, \quad (36)$$

and the shift of the inverted matrix

$$Z^{-1} \rightarrow Z^{-1} + L, \quad L = \begin{pmatrix} \lambda_E & \lambda_{H_2} \\ \lambda_{H_2} & -\lambda_{d_2} \end{pmatrix}. \quad (37)$$

The parameters λ introduced here correspond to ten H -transformations (forming $Sp(4, R)$) generalizing the Ehlers group to the EMDA system. It is worthwhile to list them separately. Parameters $\lambda_g, \lambda_e, \lambda_m$ correspond to the gravitational, electric and magnetic gauge transformations:

$$E = E_0 + \lambda_g, \quad \Phi = \Phi_0, \quad z = z_0, \quad (38)$$

$$E = E_0 - 2\lambda_e \Phi_0 - \lambda_e^2 z_0, \quad \Phi = \Phi_0 + \lambda_e z_0, \quad z = z_0, \quad (39)$$

$$E = E_0, \quad \Phi = \Phi_0 + \lambda_m, \quad z = z_0. \quad (40)$$

The scale transformation now reads

$$E = e^{2\lambda} E_0, \quad \Phi = e^\lambda \Phi_0, \quad z = z_0. \quad (41)$$

The $SL(2, R)$ S -duality subgroup is

$$\begin{aligned} E &= E_0, \quad \Phi = \Phi_0, \quad z = z_0 + \lambda_{d_1}, \\ E &= E_0 + \lambda_{d_2} \frac{\Phi_0^2}{1 + \lambda_{d_2} z_0}, \quad \Phi = \frac{\Phi_0}{1 + \lambda_{d_2} z_0}, \quad z = \frac{z_0}{1 + \lambda_{d_2} z_0}, \\ E &= E_0, \quad \Phi = e^{\lambda_{d_3}} \Phi_0, \quad z = e^{2\lambda_{d_3}} z_0. \end{aligned} \quad (42)$$

The most nontrivial part of the group in the physical terms consists of the pair of the electric and magnetic Harrison transformations

$$E = E_0, \quad \Phi = \Phi_0 + \lambda_{H_1} E_0, \quad z = z_0 - 2\lambda_{H_1} \Phi_0 - \lambda_{H_1}^2 E_0, \quad (43)$$

$$\begin{aligned} E &= \frac{E_0}{(1 + \lambda_{H_2} \Phi_0)^2 + \lambda_{H_2}^2 E_0 z_0}, \\ \Phi &= \frac{\Phi_0 (1 + \lambda_{H_2} \Phi_0) + \lambda_{H_2} E_0 z_0}{(1 + \lambda_{H_2} \Phi_0)^2 + \lambda_{H_2}^2 E_0 z_0}, \\ z &= \frac{z_0}{(1 + \lambda_{H_2} \Phi_0)^2 + \lambda_{H_2}^2 E_0 z_0}, \end{aligned} \quad (44)$$

and the Ehlers transformation

$$E = \frac{E_0}{1 + \lambda_E E_0}, \quad \Phi = \frac{\Phi_0}{1 + \lambda_E E_0}, \quad z = z_0 + \frac{\lambda_E \Phi_0^2}{1 + \lambda_E E_0}. \quad (45)$$

Now let us turn to construction of the infinite hierarchy of potentials. The advantage of the present formulation consists in the close similarity to the Kinnersley–Chitre description of the vacuum Einstein equations (rather than electrovacuum). So we can directly repeat the procedure in terms of matrix-valued quantities. The first step consists in writing down the whole system of equations as a manifestly G -covariant condition of self-duality. To this end one introduces the matrix-valued twist tensor

$$\tilde{\nabla} \Psi_B^A = \rho^{-1} F^{AC} \nabla F_{CB}, \quad (46)$$

whose existence is implied by the equation (22). Taking complex combination

$$H_{AB} = F_{AB} + i\Psi_{AB}, \quad (47)$$

one can directly check that it obeys the self-duality condition

$$\nabla H_{AB} = -i\rho^{-1} F_A^C \tilde{\nabla} H_{CB}. \quad (48)$$

Infinite hierarchy of potentials $\overset{n}{H}_{AB}$ now can be generated via recursive equations

$$\overset{n+1}{H}_{AB} = i \left(\overset{1n}{N}_{AB} + H_A^C \overset{n}{H}_{CB} \right), \quad (49)$$

where the quadratic potentials $\overset{mn}{N}_{AB}$ are introduced through the relation

$$\nabla \overset{mn}{N}_{AB} = \overset{m}{H}^*_{CA} \nabla \overset{n}{H}^C_B. \quad (50)$$

These relations are valid for all $n, m = 0, 1, 2, \dots$ and it is understood that

$${}^0 H_{AB} = i\epsilon_{AB}, \quad {}^{0n} N_{AB} = -i{}^n H_{AB}. \quad (51)$$

Although formally these relations coincide with those for the vacuum Einstein equations, an essential complication comes from the fact that now the potentials are non-commuting matrices. Hence the problem of explicit construction of finite transformations acting on the hierarchy is much more tedious (details will be given elsewhere).

Alternatively, various linear deformation problems were suggested to deal with the Einstein equations, which can be probed in the present case too. We briefly describe here the formulation of the Belinskii–Zakharov inverse scattering transform method appropriate to the EMDA system. It can be derived starting with the null-curvature representation of the equations of motion following from the symmetric space nature of three-dimensional σ -model target space. Such representation was derived in [33] in terms of the P, Q variables. Now we are in a position to give similar formulation directly in terms of the initial physical quantities entering the Eqs. (22).

Rewriting the system (22) as the modified chiral equation for the symmetric 4×4 matrix $F = F_{AB}$

$$(\rho F_{,\rho} F^{-1})_{,\rho} + (\rho F_{,z} F^{-1})_{,z} = 0, \quad (52)$$

one can obtain the corresponding Lax pair with a complex spectral parameter λ in the original Belinskii–Zkharov form:

$$D_1 \Psi = \frac{\rho U - \lambda V}{\rho^2 + \lambda^2} \Psi, \quad D_2 \Psi = \frac{\rho V + \lambda U}{\rho^2 + \lambda^2} \Psi. \quad (53)$$

Here $V = \rho F_{,\rho} F^{-1}$, $U = \rho F_{,z} F^{-1}$, Ψ is a matrix "wave function", and

$$D_1 = \partial_z - \frac{2\lambda^2}{\rho^2 + \lambda^2} \partial_\lambda, \quad D_2 = \partial_\rho + \frac{2\lambda\rho}{\rho^2 + \lambda^2} \partial_\lambda \quad (54)$$

are commuting operators; then the nonlinear system (52) can be regarded as the compatibility condition of the linear system $[D_1, D_2]\Psi = 0$. Similar linear system may be written in terms of the dualized variables, i.e. the matrix M satisfying Eq. (33).

An infinite algebra of the Geroch–Kinnersley–Chitre group can be obtained from here via an expansion of the linear system in power series in terms of the complex spectral parameter λ . Practical use of the 4×4 matrix Lax pair requires further study. We note another 4×4 linear problem (with different group structure) discussed recently [37].

Acknowledgements

The author thanks the organizers of this Conference for kind hospitality and support. Useful discussions with L. Bordag are gratefully acknowledged. At different stages of this research the author profited much from conversations with G.A. Alekseev, G. Neugebauer, D. Kramer, and J.B. Griffiths. The research was supported in part by the Russian Foundation for Fundamental Research Grant 93-02-16977, the International Science Foundation and Russian Gov. Grant M79300 and the INTAS Grant 93-3262.

References

- [1] R. Geroch, Journ. Math. Phys. **13**, 394 (1972).
- [2] J. Ehlers, in *Les Theories Relativistes de la Gravitation*, CNRS, Paris, p. 275, 1959.
- [3] B. K. Harrison, J. Math. Phys. **9**, 1774 (1968).
- [4] W. Kinnersley, Journ. Math. Phys. **14**, 651 (1973); **18**, 1529 (1977).
- [5] W. Kinnersley and D. Chitre, Journ. Math. Phys. **18**, 1538 (1977); **19**, 1926, 2037 (1978).
- [6] G. Neugebauer, Habilitationschrift, Jena, 1969, (unpublished).
- [7] G. Neugebauer and D. Kramer, Ann. der Physik (Leipzig) **24**, 62 (1969).
- [8] F.J. Ernst, Phys. Rev. **167** (1968) 1175; **168** (1968) 1415.
- [9] B.K. Harrison, Phys. Rev. Lett., **41**, 1197, (1978).
- [10] D. Maison, Journ. Math. Phys. **21**, 871 (1979).
- [11] V.A. Belinskii and V.E. Zakharov. Sov. Phys. JETP, **48**, 985 (1978); **50**, 1 (1979).
- [12] I. Hauser and F. J. Ernst, Phys. Rev. **D 20**, 362, 1783 (1979).
- [13] G. Neugebauer, Journ. Phys. A: **12**, L67; **1**, L19 (1979).
- [14] G.A. Alekseev, Pis'ma Zh. Eksp. Teor. Fiz. **32**, 301 (1980).
- [15] C. Cosgrove, Journ. Math. Phys., **21**, 2417 (1981); **22**, 2624 (1981); **23**, 615 (1982).
- [16] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, Exact Solutions of the Einstein's field equations, CUP, 1980.
- [17] A. Eris, M. Gürses, and A. Karasu, Journ. Math. Phys. **25**, 1489 (1984); M. Gürses, in [18], p. 199.
- [18] "Exact solutions of Einstein's equations: Techniques and results." Ed. W. Dietz and C. Hoenselaers. Lecture Notes in Physics, **205**, 1985.
- [19] D. Kramer in *Unified field theories of more than four dimensions*, Eds. V. De Sabbata and E. Schmutzter, WS, 1983, p. 248; G. Neugebauer and D. Kramer, in *Galaxies, Axisymmetric systems and Relativity*, ed. by M. MacCallum, CUP, 1986, p.149; D. Kramer and G. Neugebauer in [18], p. 1.
- [20] P. Breitenlohner and D. Maison, in [18], p. 276; Ann. IHP **46**, p. 215, (1987).
- [21] P. S. Letelier, Journ. Math. Phys., **23**, 467. (1985).
- [22] G.A. Alekseev, in Trudi MIAN , **176**, (1987), p. 211.
- [23] N.R. Sibgatullin, Oscillations and waves in strong gravitational fields, Moscow, Nauka, 1984.

- [24] R.K. Dodd, in *Soliton Theory, a survey of results*, Eds. A.P. Fordy, Manchester UP, 1990, p.174.
- [25] E. Verdaguer, Phys. Repts, **229**, 2, (1993)
- [26] J. B. Griffiths, Colliding Plane Waves in General Relativity, Clarendon Press 1991.
- [27] D. Maison, Gen. Rel. and Grav. **10**, 717 (1979); V. Belinskii and R. Ruffini, Phys. Lett. **B89**, 195 (1980).
- [28] B. Clement, Gen. Rel. and Grav., **18**, 137, 861, (1986); Phys. Lett. **A 118**, 11, (1986).
- [29] B. Julia, in: “*Proceedings of the John Hopkins Workshop on Particle Theory*”, Baltimore (1981); in “*Superspace and Supergravity*”, Eds. S. Hawking and M. Rocek, Cambridge, 1981; in “*Unified Theories of More than Four Dimensions*” Eds. V. De Sabbata and E. Shmutzter, WS, Singapore 1983.
- [30] P. Breitenlohner, D. Maison, and G. Gibbons, Comm. Math. Phys. **120**, 253 (1988).
- [31] H. Nicolai, Phys. Letts. **B194** (1987) 402.
- [32] D.V. Gal’tsov and O.V. Kechkin, Phys. Rev. **D50** (1994) 7394; (hep-th/9407155).
- [33] D.V. Gal’tsov, Phys. Rev. Lett. **74** (1995) 2863, (hep-th/9410217).
- [34] I. Bakas, *Nucl. Phys.* **B428** (1994) 374; *Phys. Lett.* **B343** (1995) 103; J. Maharana, *Hidden Symmetries of Two Dimensional String Effective Action*, hep-th/9502001; *Symmetries of the Dimensionally Reduced String Effective Action*, hep-th/9502002; A. Sen, *Duality Symmetry Group of Two Dimensional Heterotic String Theory*, preprint TIFR-TH-95-10, hep-th/9503057. J.H. Schwarz, *Classical Symmetries of Some Two-Dimensional Models*, CALT-68-1978, hep-th/9503078.
- [35] D.V. Gal’tsov and O.V. Kechkin, *U-Duality and Symplectic Formulation of Dilaton-Axion gravity*, Preprint IC/95/155, hep-th/9507005; Gal’tsov D.V., Garcia A.A., Kechkin O.V., J. Math. Phys., 1995, v. 36, n. 9, p. 1 – 19, Gal’tsov D.V., Kechkin O.V., *Matrix Dilaton–Axion for Heterotic String in Three Dimensions*, preprint DTP–MSU 95/24, hep-th/9507164.
- [36] D.V. Gal’tsov, O.V. Kechkin, *Hidden symmetries in dilaton–axion gravity*, in “*Geometry and Integrable Models*”, P.N. Pyatov and S.N. Solodukhin (Eds.), World Scientific, 1996.
- [37] L.A. Bordag, A.B. Yanovski, J. Phys. **A 28**, 4007, (1995).